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Abstract

This article deals with a mean-field model. We consider a large number of particles interacting through their empirical law. We know that there is a unique invariant probability for this diffusion. We look at functional inequalities. In particular, we briefly show that the diffusion satisfies a Poincaré inequality. Then, we establish a so-called WJ-inequality, which is independent from the number of particles.

Key words and phrases: Mean-field model ; Poincaré inequality ; Transportation inequality ; High dimension

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1 Introduction

1.1 Model

First, we consider a sequence $(X_0^i)_{i\geq 1}$ of independent and identically distributed random variables with common law μ_0 on $\mathbb{R}^d$. We also consider a sequence of independent Brownian motions $(B^i)_{i\geq 1}$ on $\mathbb{R}^d$. These Brownian motions are assumed to be independent from the previously introduced random variables.

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The system of interacting particles that we look at evolves in the landscape of a potential V. This potential is denoted as the confining potential. Its effect is roughly speaking to locate the particles in a compact of $\mathbb{R}^d$. We assume that the confining potential V is convex at infinity but non-globally convex. However, we assume that the Hessian of V is minorated: $\nabla^2 V \geq -\theta \mathrm{Id}$, where θ is a positive constant.

We now introduce the so-called interacting potential F. We do not assume that it is either convex or not convex. However, the following inequality is required: $\nabla^2 F \geq -\alpha \mathrm{Id}$.

The precise assumptions are given subsequently. We consider the system of interacting particles

$$\begin{cases} X_t^{i,N} = X_0^i + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N}\right) ds - \int_0^t \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_s^{i,N} - X_s^{j,N}\right) ds \\ 1 \leq i \leq N \end{cases} . \tag{1}$$

Here, N is large.

This system of N particles in $\mathbb{R}^d$ may be seen as one particle in $\mathbb{R}^{dN}$. Indeed, let us introduce:

$$\mathcal{X}_t^N := \left(X_t^{1,N}, \cdots, X_t^{N,N}\right) \in \mathbb{R}^{dN} .$$

Thus, the diffusion $\mathcal{X}$ is a simple diffusion evolving in the landscape of a potential of $\mathbb{R}^{dN}$:

$$\mathcal{X}_t^N = \mathcal{X}_0^N + \sigma \mathcal{B}_t - N \int_0^t \nabla \Upsilon^N \left(\mathcal{X}_s^N\right) ds \,,$$

with $\mathcal{B} := \left(B^1, \cdots, B^N\right)$ and where the potential Υ^N is defined as follows

$$\Upsilon^N \left(X_1, \cdots, X_N\right) := \frac{1}{N}\sum_{i=1}^N V(X_i) + \frac{1}{2N^2}\sum_{i=1}^N \sum_{j=1}^N F\left(X_i - X_j\right) . \tag{2}$$

This model has a natural application in the financial markets in which there is a huge number of agents who act in function of the global behaviour of the system. We can also think at the system of exchanges between banks, a classical result being the importance of the interaction in the probability of bankrupt.

With some other hypotheses, these equations are used as a model for the growth of cerebral tumors. Indeed, each of the cells tries to be alone for developping itself.

Mean-field system of particles are also used in social interaction, [CDPS10]. They also appeared quite naturally in the study of stochastic partial differential equation, see [CX10].

1.2 Well-known results

1.2.1 Existence

Basically, we assume the following hypotheses

- The potential V is convex at infinity. In fact, we even have: $\nabla^2 V(\infty) = \infty$.

- In the neighbourhood of infinity, the confining potential behaves like a polynomial function: $\langle \nabla V(x); x\rangle = ||x||^{2m} + o\left(||x||^{2m}\right)$ at infinity, where $m \in \mathbb{N}$.
- In the neighbourhood of infinity, the interacting potential behaves like a polynomial function: $\langle \nabla F(x); x\rangle = ||x||^{2n} + o\left(||x||^{2n}\right)$ at infinity, where $n \in \mathbb{N}$.

Then, we consider the maximal degree by putting $q := \max\{m; n\}$. Thus, if the initial law admits a moment of order $2q$ that is to say if

$$\int_{\mathbb{R}^d} ||x||^{2q} \mu_0(dx) < \infty\,,$$

there exists a unique solution to Equation (1). We denote by $\left(\mathcal{X}_t^N\right)_{t\geq 0}$ this diffusion solution. Let us remark that the existence of a solution does not depend on the number N of particles.

1.2.2 Invariant probability

There is a unique invariant probability $\mu^{\sigma,N}$ on $\mathbb{R}^{dN}$ for Diffusion (1):

$$\mu^{\sigma,N}(d\mathcal{X}) := Z_{\sigma,N}^{-1} \exp\left\{-\frac{2N}{\sigma^2}\Upsilon^N(\mathcal{X})\right\} d\mathcal{X}\,,$$

where the potential Υ^N is defined in (2).

This potential has sense when N goes to infinity. Indeed, it represents the energy associated to the probability $\frac{1}{N}\sum_{i=1}^N \delta_{X_i}$.
We can observe that there is an N in factor so that the invariant probability and the long-time behavior does depend on N.

1.2.3 Long-time behavior

Thanks to Bakry, Barthe, Cattiaux and Guillin, see [BBCG08], the measure $\mu^{\sigma,N}$ satisfies a Poincaré inequality:

$$\mathrm{Var}_{\mu^{\sigma,N}}(f) \leq \frac{1}{C_\sigma(N)} \int_{\mathbb{R}^{dN}} ||\nabla f||^2\, d\mu^{\sigma,N}$$

for any f which is a smooth function from $\mathbb{R}^{dN}$ to $\mathbb{R}$. This inequality is equivalent to the convergence inequality

$$\left|\left|\mathcal{P}_t^N f - \mathbb{E}_{\mu^{\sigma,N}}(f)\right|\right|_2^2 \leq \exp\left\{-\frac{2}{C_\sigma(N)}t\right\} \mathrm{Var}_{\mu^{\sigma,N}}(f)\,,$$

where we have put $\mathcal{P}_t^N f(x) := \mathbb{E}_x\left\{f\left(\mathcal{X}_t^N\right)\right\}$.

Let us point out that the constant $C_\sigma(N)$ which intervenes in the inequality does not have any reason to be independent from the number of particles, that is to say from the dimension of the space in which evolves the solution $\mathcal{X}^N$.

1.2.4 Hydrodynamical limit

Intuitively, $(X_t^{i,N})_{t\geq 0}$ behaves like the diffusion $(\overline{X}_t^i)_{t\geq 0}$ when N is large. Here, the diffusion $\overline{X}^i$ is defined by the equation:

$$\overline{X}_t^i = X_0^i + \sigma B_t^i - \int_0^t \nabla V\left(\overline{X}_s^i\right) ds - \int_0^t \nabla F * \mu_s\left(\overline{X}_s^i\right) ds$$

with $\mu_s := \mathbb{P}_{\overline{X}_s^1}$. Indeed, we can observe that the influence of the particle j on the particle 1 becomes small when N is large. So, roughly speaking, the particles of the interacting system of particles become independent. However, Equation (1) can be written as follows

$$X_t^{i,N} = X_0^i + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N}\right) ds - \int_0^t \nabla F * \mu_s^N\left(X_s^{i,N}\right) ds\,,$$

with $\mu_s^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_s^{j,N}}$. If the particles become independent, the measure μ_s^N converges to μ_s which explains why the particles $X^{i,N}$ intuitively are closed to $\overline{X}^i$.

We now assume $\mathbb{E}\left\{||X_0^1||^{8q^2}\right\} < \infty$ that is to say

$$\int_{\mathbb{R}^d} ||x||^{8q^2}\mu_0(dx) < \infty\,,$$

where we remind the reader that q is the maximum of the degrees of V and F. Under this assumption, we have a so-called propagation of chaos:

$$\sup_{t\in[0;T]} \mathbb{E}\left\{\left|\left|X_t^{i,N} - \overline{X_t^i}\right|\right|^2\right\} \longrightarrow 0$$

for any $T > 0$.

See [Mél96, Szn91, McK67, McK66, BRTV98, BAZ99] for review on propagation of chaos.

1.3 Aim of the article

In this paper, we aim to show that there is a rate of convergence of the diffusion $\mathcal{X}^N$ to the unique invariant probability and that this rate of convergence is uniform with respect to the number of particles.

1.4 Outline of the article

First, we present the hypotheses of the article. In the next section, we justify why the diffusion satisfies a Poincaré inequality. Then, we give some classical results on this inequality and we discuss about an eventual uniform Poincaré inequality. In section three, we present the framework of the paper. Finally, in a last section, we give the main result and its proof.

1.5 Assumptions

Let us present the assumptions of the paper.

(A-1) V is a smooth function on $\mathbb{R}^d$.

(A-2) V is convex at infinity: for any $\lambda > 0$, there exists $R_\lambda > 0$ such that $\nabla^2 V(x) > \lambda \mathrm{Id}$ for any $||x|| \geq R_\lambda$.

(A-3) There exists a convex nonnegative function V_0 such that $\nabla^2 V_0(0) = 0$ and $V(x) = V_0(x) - \frac{\theta}{2}||x||^2$, with $\theta > 0$.

(A-4) There exist $m \in \mathbb{N}^*$ and $C > 0$ such that $||\nabla V(x)|| \leq C\left(1 + ||x||^{2m-1}\right)$ for all $x \in \mathbb{R}^d$.

(A-5) It holds $|V(x)| \leq c\|x\|^2$ for $\|x\| \leq 1$, in particular, $V(0) = 0$.

(A-6) $F(x) = G(||x||) - \frac{\alpha}{2}||x||^2$, where G is a polynomial and even function with degree equal to $\deg(G) =: 2n \geq 2$ and $G(0) = 0$. Here, α is not necessarily positive.

(A-7) $\int_{\mathbb{R}^d} \|x\|^{8q^2} \mu_0(dx) < \infty$ with $q := \max\{m; n\}$.

(A-8) The entropy of the probability measure is finite. In other words, μ_0 is absolutely continuous with respect to the Lebesgue measure and we have $\int_{\mathbb{R}^d} u_0(x) \log(u_0(x)) dx < \infty$ where u_0 is the density of μ_0.

2 Preliminaries

We begin by looking at Poincaré inequality for our model. Let us remind the reader that the invariant probability $\mu^{\sigma,N}$ is of the form e^{-U}. According to [BBCG08], it satisfies a Poincaré inequality under simple hypotheses. Indeed:

Proposition 2.1. *Let k be a positive integer. Let $\mu(dx) := e^{-U(x)}dx$ be a probability measure on $\mathbb{R}^k$. We assume that the potential U is $\mathcal{C}^2$ and bounded from below. If there exist $\alpha > 0$ and $R \geq 0$ such that for $|x| \geq R$,*

$$\langle x; \nabla U(x) \rangle \geq \alpha |x| \,,$$

then μ satisfies a Poincaré inequality with constant

$$\frac{4\left(1 + \left(\exp\left[\frac{1}{2}\left(\alpha R + 1 - k\right)\right] + 1\right)\kappa_R\right)}{\left(\alpha - \frac{k-1}{R}\right)^2}$$

for any R such that $\alpha - \frac{k-1}{R} > 0$. Here, κ_R is the Poincaré constant of μ restricted to the ball $\mathbb{B}(0; R)$.

The proof is omitted, see [BBCG08]. The global idea is the following. We can apply Theorem 1.4 in [BBCG08]. Indeed, we consider a sequence of smooth functions W_n which satisfies

- $W_n(x) = \exp\left\{\frac{1}{2}\left(\alpha - \frac{k-1}{R}\right)|x|\right\}$ for $|x| \geq R$.
- $W_n(x) = \exp\left\{\frac{1}{2}\left(\alpha R + 1 - k - \frac{1}{n^2}\right)\right\}$ for $|x| \leq R - \frac{1}{n}$.

Consequently, we have the inequality

$$\begin{aligned}
&\Delta W_n(x) - \langle \nabla W_n(x); \nabla U(x)\rangle \\
\leq &- \frac{1}{4}\left(\alpha - \frac{k-1}{R}\right)^2 + \left(\exp\left\{\frac{1}{2}\left(\alpha R + 1 - k - \frac{1}{n^2}\right)\right\} + 1\right) \mathbb{1}_{\mathbb{B}(0;R)}(x) .
\end{aligned}$$

We may apply this proposition to our model under the set of assumptions (A-1)–(A-7).

We now give two classical results about functional inequalities.

Proposition 2.2. *Let μ be a probability measure on $\mathbb{R}^k$ and U be a bounded function from $\mathbb{R}^k$ to $\mathbb{R}$. We define the measure ν – the perturbation of μ by U – as follows*

$$d\nu := \frac{e^U}{Z} d\mu \quad \textit{with} \quad Z := \int e^U d\mu .$$

If μ satisfies a Poincaré inequality with constant C then ν satisfies a Poincaré inequality with constant $C \exp\left\{\sup_{\mathbb{R}^k} U - \inf_{\mathbb{R}^k} U\right\}$.

Proof. Let f be any smooth function. For $a = \int_{\mathbb{R}^k} f(y) d\mu(y)$ we have:

$$\begin{aligned}
&\int_{\mathbb{R}^k} (f(x) - a)^2 \, d\nu(x) \\
&= \frac{1}{Z} \int_{\mathbb{R}^k} (f(x) - a)^2 \, e^{U(x)} d\mu(x) \\
&\leq \frac{1}{Z} \exp\left\{\sup_{\mathbb{R}^k} U\right\} \int_{\mathbb{R}^k} (f(x) - a)^2 \, d\mu(x) \\
&\leq \frac{1}{Z} \exp\left\{\sup_{\mathbb{R}^k} U\right\} \int_{\mathbb{R}^k} ||\nabla f(x)||^2 \, d\mu(x) \\
&\leq \exp\left\{\sup_{\mathbb{R}^k} U\right\} \int_{\mathbb{R}^k} ||\nabla f(x)||^2 \, e^{-U(x)} d\nu(x) \\
&\leq \exp\left\{\sup_{\mathbb{R}^k} U - \inf_{\mathbb{R}^k} U\right\} \int_{\mathbb{R}^k} ||\nabla f(x)||^2 \, d\nu(x) .
\end{aligned}$$

As $\mathrm{Var}_\nu(f)$ is the infimum of

$$\int_{\mathbb{R}^k} (f(x) - a)^2 \, d\nu(x)$$

as $a \in \mathbb{R}^k$, this achieves the proof. □

The other well-known result is the tensorization one. We present it in $\mathbb{R}^2$ without any loss of generality.

Proposition 2.3. *Let μ_1 and μ_2 be two probability measures on $\mathbb{R}$. We assume that both probability measures μ_1 and μ_2 satisfies a Poincaré inequality with constant C. Then, the probability measure on $\mathbb{R}^2$, $\mu_1 \otimes \mu_2$, satisfies a Poincaré inequality with constant C.*

Proof. For any smooth function from $\mathbb{R}^2$ to $\mathbb{R}$, one can easily prove the inequality

$$\mathrm{Var}_{\mu_1\otimes\mu_2}(f) \leq \mathbb{E}_{\mu_1\otimes\mu_2}\left(\mathrm{Var}_{\mu_1}(f)\right) + \mathbb{E}_{\mu_1\otimes\mu_2}\left(\mathrm{Var}_{\mu_2}(f)\right)\,,$$

where the notation Var_{μ_1} (respectively Var_{μ_2}) means that the first (respectively the second) variable is the only one to be affected by the integration. Indeed, this inequality is equivalent to

$$\mathbb{E}_{\mu_1\otimes\mu_2}(f^2) - \mathbb{E}_{\mu_1\otimes\mu_2}\left[\left(\mathbb{E}_{\mu_1}(f)\right)^2\right] - \mathbb{E}_{\mu_1\otimes\mu_2}\left[\left(\mathbb{E}_{\mu_2}(f)\right)^2\right] + \left(\mathbb{E}_{\mu_1\otimes\mu_2}(f)\right)^2 \geq 0\,,$$

which can also be written as follows

$$\mathbb{E}_{\mu_1\otimes\mu_2}\left[\left(f - \mathbb{E}_{\mu_1}(f) - \mathbb{E}_{\mu_2}(f) + \mathbb{E}_{\mu_1\otimes\mu_2}(f)\right)^2\right] \geq 0\,.$$

However, Poincaré inequality implies

$$\mathrm{Var}_{\mu_1}(f) \leq C\int_{\mathbb{R}} |\nabla f|^2\, d\mu_1$$

and

$$\mathrm{Var}_{\mu_2}(f) \leq C\int_{\mathbb{R}} |\nabla f|^2\, d\mu_2\,,$$

which achieves the proof. □

We proceed in a similar way for the tensorization of k measures on $\mathbb{R}$.

In the model that we consider we have a Poincaré inequality. However, the constant may depend on the dimension.

The constant which appears in Proposition 2.1 does depend on the dimension:

$$\frac{4\left(1 + \left(\exp\left[\frac{1}{2}\left(\alpha R + 1 - k\right)\right] + 1\right)\kappa_R\right)}{\left(\alpha - \frac{k-1}{R}\right)^2}\,.$$

Indeed, R has to be such that $\alpha > \frac{k-1}{R}$ which means $R > \frac{k-1}{\alpha}$. Consequently, the constant is more than

$$\frac{4\left(1 + 2\kappa_{\frac{k-1}{R}}\right)}{\left(\alpha - \frac{k-1}{R}\right)^2}\,.$$

However, we can remark $\lim\limits_{R\to\infty} \kappa_R = +\infty$.

By using tensorization result, we prove easily that the measure

$$\exp\left\{-\frac{2}{\sigma^2}\sum_{k=1}^{N} V(x_k)\right\} dx_1\cdots dx_N$$

satisfies a Poincaré inequality with a constant which does not depend on the dimension N. If we assume that F is bounded, we can use the perturbation result to prove that the measure $\exp\left\{-\frac{2}{\sigma^2}N\Upsilon_N(x_1,\cdots,x_N)\right\}dx_1\cdots dx_N$ satisfies a Poincaré inequality. However, the constant just obtained does depend on the dimension.

Let us remark that we can write

$$N\Upsilon^N(x_1,\cdots,x_N)=\sum_{k=1}^N\left(V(x_k)+F*\eta^{\mathcal{X}}(x_k)\right)\,,$$

with $\eta^{\mathcal{X}}:=\frac{1}{N}\sum_{k=1}^N\delta_{x_k}$. However, we can not use the tensorization result. Intuitively, the propagation of chaos means that the particles become independent so that we have a Poincaré inequality with a constant which does not depend on the dimension.

In the following, we deal with WJ-inequality to get inequality independent from the dimension.

3 Framework

Let us give the framework (definitions and basic propositions) of the current work. We begin by introducing the Wasserstein distance.

Definition 3.1. *For any probability measures on $\mathbb{R}^d$, μ and ν, the Wasserstein distance between μ and ν is*

$$\mathbb{W}_2\left(\mu\,;\,\nu\right):=\sqrt{\inf\ \mathbb{E}\left\{\left|\left|X-Y\right|\right|^2\right\}},$$

where the infimum is taken over the random variables X and Y with law μ and ν respectively.

The Wasserstein distance can be characterized in the following way, thanks to Brenier's theorem, see [Bre91].

Proposition 3.2. *Let μ and ν be two probability measures on $\mathbb{R}^d$. If μ is absolutely continuous with respect to the Lebesgue measure, there exists a convex function τ from $\mathbb{R}^d$ to $\mathbb{R}$ such that the following equality occurs for every bounded test function g:*

$$\int_{\mathbb{R}^d}g(x)\nu(dx)=\int_{\mathbb{R}^d}g\left(\nabla\tau(x)\right)\mu(dx)\,.$$

Then, we write

$$\nu=\nabla\tau\#\mu\,,$$

and we have the following equality

$$\mathbb{W}_2\left(\mu\,;\,\nu\right)=\sqrt{\int_{\mathbb{R}^d}\left|\left|x-\nabla\tau(x)\right|\right|^2\mu(dx)}\,.$$

The key-idea of the paper is a so-called $WJ_{V,F}$-inequality. Let us present the expression that we denote by $J_{V,F}(\nu \mid \mu)$ if μ is absolutely continuous with respect to the Lebesgue measure:

$$
\begin{aligned}
&J_{V,F}\left(\nu \mid \mu\right) := J_{V,0}\left(\nu \mid \mu\right) \\
&+ \frac{1}{2}\iint_{\mathbb{R}^{2d}} \left\langle \nabla F\left(\xi(x,y)\right) - \nabla F(x-y)\,;\, \xi(x,y) - (x-y)\right\rangle \mu(dx)\mu(dy)\,,
\end{aligned}
$$

with $\xi(x,y) := \nabla\tau(x) - \nabla\tau(y)$ and

$$
\begin{aligned}
J_{V,0}\left(\nu \mid \mu\right) &:= \frac{\sigma^2}{2}\int_{\mathbb{R}^d} \Big(\Delta\tau(x) + \Delta\tau^*\left(\nabla\tau(x)\right) - 2d\Big)\mu(dx) \\
&+ \int_{\mathbb{R}^d} \left\langle \nabla V\left(\nabla\tau(x)\right) - \nabla V(x)\,;\, \nabla\tau(x) - x\right\rangle \mu(dx)\,,
\end{aligned}
$$

where τ^* denotes the Legendre transform of τ. Here, we have $\nu = \nabla\tau\#\mu$. We now present the transportation inequality, already used in [AGS08, BGG12, BGG13, SvR05], on which the article is based.

Definition 3.3. *Let μ be a probability measure on $\mathbb{R}^d$ absolutely continuous with respect to the Lebesgue measure and $C > 0$. We say that μ satisfies a $WJ_{V,F}(C)$-inequality if the inequality*

$$
C\mathbb{W}_2^2\left(\nu\,;\,\mu\right) \leq J_{V,F}\left(\nu \mid \mu\right)
$$

holds for any probability measure ν on $\mathbb{R}^d$.

In the following, we aim to establish WJ-inequality for the invariant probability $\mu^{\sigma,N}$ of Diffusion (1). It is well known that $\mu^{\sigma,N}$ is absolutely continuous with respect to the Lebesgue measure. Consequently, we can apply Brenier's theorem. So, the $WJ_{V,F}$-inequality reduces to an inequality on the convex functions τ from $\mathbb{R}^d$ to $\mathbb{R}$.

4 Main results

We here use the result from [BGG12]. We know that $WJ_{V,0}$-inequality holds with some constant $C_{\sigma,N}$. Therefore $WJ_{V,F}$-inequality holds with the constant $C_{\sigma,N} - (\alpha+\theta)$:

Proposition 4.1. *Under the assumptions (A-1)–(A-7), a $WJ_{V,F}$ inequality holds for the measure $\mu^{\sigma,N}$ with the constant $C_{\sigma,N} - (\alpha+\theta)$, where*

$$
C_{\sigma,N} := \max_{R>0} C_\sigma(N,R)
$$

and

$$
C_\sigma(N,R) := \min\left\{ \frac{K(R)}{3};\ \frac{\sigma^2}{72R^2} e^{-\frac{2}{\sigma^2}S(R)};\ \frac{K(R)}{3}\frac{3^{dN}-2^{dN}}{2^{dN}} e^{\frac{2}{\sigma^2}(I(R)-S(R))} \right\}, \tag{3}
$$

with

$$K(R) := \inf_{\|\chi\|\geq R} N\nabla^2\Upsilon_0^N(\chi), \tag{4}$$

$$I(R) := \inf_{\|\chi\|\leq 2R} N\Upsilon^N(\chi), \tag{5}$$

$$S(R) := \sup_{\|\chi\|\leq 3R} N\Upsilon^N(\chi). \tag{6}$$

In definition of $K(R)$, the infimum is understood as the smallest eigenvalue of $N\nabla^2\Upsilon_0^N(\chi)$. Furthermore, Υ_0^N is defined as follows

$$N\Upsilon_0^N(\chi) := \sum_{i=1}^N V_0(\chi_i) + \frac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N G(||\chi_i - \chi_j||)\,.$$

The proof is omitted and consists in following carefully [BGG12].

However, nothing ensures us *a priori* that $C_{\sigma,N} - (\alpha+\theta)$ is positive. This is the aim of next theorem:

Theorem 4.2. *There exists $\widehat{\sigma_c}$ such that $C_{\sigma,N} > \alpha+\theta$ for any N if $\sigma \geq \widehat{\sigma_c}$.*

Proof. We will use Proposition 4.1 and notation introduced therein. From (A2) and (A3),

$$\nabla^2 N\Upsilon_0^N(\chi) \geq (12\alpha + 12\theta + 12)\mathrm{Id}, \qquad \text{if } \|\chi\| \geq R := R_{12\alpha+11\theta+12}.$$

Therefore

$$\frac{K(R)}{3} \geq 4(\alpha+\theta+1). \tag{7}$$

From (A4) and (A5) it follows that

$$|V(x)| \leq C\left(\|x\| + \frac{\|x\|^{2m}}{2m}\right), \qquad x\in\mathbb{R}^d,$$

which together with (A5) implies that

$$|V(x)| \leq c\left(\|x\|^2 + \|x\|^{2m}\right), \qquad x\in\mathbb{R}^d. \tag{8}$$

On the other hand, from (A6), $F(x) = G(\|x\|) - \frac{\alpha}{2}\|x\|^2 = \sum_{k=1}^n b_{2k}\|x\|^{2k}$ for some $b_{2k}\in\mathbb{R}$. This and (8) give

$$\begin{aligned} N\Upsilon^N(\chi) &= \sum_{i=1}^N V(\chi_i) + \tfrac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N F(\chi_i-\chi_j) \\ &\leq c\sum_{i=1}^N\left(\|\chi_i\|^2 + \|\chi_i\|^{2m}\right) + \tfrac{1}{2N}\sum_{i=1}^N\sum_{j=1}^N\sum_{k=1}^n b_{2k}\|\chi_i-\chi_j\|^{2k}. \end{aligned} \tag{9}$$

We continue using elementary inequality $|a-b|^{2k} \le 2^{2k-1}(|a|^{2k}+|b|^{2k})$

$$\begin{aligned} N\Upsilon^N(\chi) &\le c\sum_{i=1}^N \left(\|\chi_i\|^2 + \|\chi_i\|^{2m}\right) + \sum_{k=1}^n b_{2k}2^{2k-1}\sum_{i=1}^N \|\chi_i\|^{2k} \\ &\le c\left(9R^2 + 9^m R^{2m}\right) + \sum_{k=1}^n b_{2k}2^{2k-1}9^k R^{2k} \le c' R^{2q}, \qquad \text{if } \|\chi\| \le 3R, \end{aligned}$$

here $q = \max\{m,n\}$ and the constant c' depends only on V and F. Thus

$$S(R) \le c' R^{2q}. \tag{10}$$

Finally, to estimate $I(R)$, we use (A3) and (A6). For $\|\chi\| \le 2R$,

$$\begin{aligned} N\Upsilon^N(\chi) &\ge -\tfrac{\theta}{2}\sum_{i=1}^N \|\chi_i\|^2 - \tfrac{\alpha}{4N}\sum_{i=1}^N\sum_{j=1}^N \|\chi_i - \chi_j\|^2 \\ &\ge -2\theta R^2 - \tfrac{\alpha}{4N}\sum_{i=1}^N\sum_{j=1}^N 2\left(\|\chi_i\|^2 + \|\chi_j\|^2\right) \ge -(2\theta+4\alpha)R^2, \end{aligned}$$

hence

$$I(R) \ge -(2\theta + 4\alpha)R^2. \tag{11}$$

We are ready to estimate $C_\sigma(N,R)$. The first term in the minimum on the right hand side of (3) is greater than $4(\alpha+\theta+1)$, see (7). The second term is greater than

$$\frac{\sigma^2}{72R^2} e^{-\frac{2}{\sigma^2}c' R^{2q}},$$

see (10), which in turn is larger than $\alpha+\theta+1$ for σ large enough. Finally, the third term in the minimum in (3) is greater than

$$4(\alpha+\theta+1)\frac{1}{2}e^{\frac{2}{\sigma^2}(-(2\theta+4\alpha)R^2 - c' R^{2q})} \ge \alpha+\theta+1,$$

for large σ. To summarise, $C_\sigma(N,R) \ge \alpha+\theta+1$ and, consequently,

$$C_{\sigma,N} \ge \alpha+\theta+1$$

for every $N \in \mathbb{N}$ and for large σ. □

Consequently, if σ is large enough, $WJ_{V,F}$-inequality holds with a constant which does not depend on the dimension.

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